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LETTER TO THE EDITOR

On some quantum R matrices associated with representations of $U_q(sl(2,\mathbb{C}))$ when q is a root of unity

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Abstract. We construct quantum R matrices associated with certain non-cyclic irreducible representations of $U_q(sl(2,\mathbb{C}))$ at roots of unity which are parametrized by one continuous parameter. We find solutions for the two- and three-dimensional representations via Baxterization. The two-dimensional case corresponds to the free fermion model of Fan and Wu.

An interesting problem from the point of view of solvable statistical models is the construction of quantum R matrices associated with representations of quantized universal enveloping algebras at roots of unity. Recently, quantum R matrices associated with cyclic representations of $U_q(sl(2,\mathbb{C}))$ for $q^m=1$ (m odd) and $U_q(sl(3,\mathbb{C}))$ for $q^3=1$ were constructed [1-3]; in both cases only odd-dimensional representations were examined. In this letter, for a given irreducible representation at root of unity, our approach to constructing the quantum R matrix consists of first solving the associated solution of the braid relation (6); the quantum R matrix is then obtained through Baxterization. We consider the case of $U_q(sl(2,\mathbb{C}))$ and restrict our discussion to representation of the highest-weight type (non-cyclic) which are parametrized by one continuous parameter μ; we solve for the two- and three-dimensional representations. It turns out that the two-dimensional case leads to the free fermion model of Fan and Wu [4]; to our knowledge this connection is new. Let us now briefly introduce $U_q(sl(2,\mathbb{C}))$ and the representations considered.

$U_q(sl(2,\mathbb{C}))$ is an associative $\mathbb{C}$-algebra generated by four generators J^+, J^-, k and k^{-1} which satisfy the following relations

$$kk^{-1}=k^{-1}k=1 \qquad kJ^{\pm}k^{-}=q^{\pm}J^{\pm} \qquad [J^+,J^-]=\frac{k^2-k^{-2}}{q-q^{-1}} \tag{1a}$$

in terms of the usual element H of the Cartan subalgebra

$$k=q^{H/2} \qquad [H,J^{\pm}]=\pm 2J^{\pm}. \tag{1b}$$

$U_q(sl(2,\mathbb{C}))$ is a Hopf algebra with comultiplication Δ, antipode S and co-unit ε defined as follows

$$\begin{aligned}&\Delta(J^{\pm})=J^{\pm}\otimes k+k^{-}\otimes J^{\pm} \qquad S(J^{\pm})=-q^{\mp}J^{\pm} \qquad S(k)=k^{-1}\\ &\varepsilon(J^{\pm})=0 \qquad \varepsilon(k)=1.\end{aligned} \tag{1c}$$

Let V be an N-dimensional vector space whose basis we denote by v_i $0 \leq i \leq N-1$. Roche and Arnaudon [5] propose the following highest-weight representation (we make the following change of variable: $\mu \to \mu + N - 1$)

$$\begin{aligned} &kv_i = q^{(\mu+N-1)/2-i} v_i \qquad J^+ v_0 = 0 \qquad J^- v_{N-1} = 0 \\ &J^+ v_i = [(i)_q (\mu + N - i)_q]^{1/2} v_{i-1} \qquad 1 \leq i \leq N-1 \\ &J^- v_i = [(i+1)_q (\mu + N - 1 - i)_q]^{1/2} v_{i+1} \qquad 0 \leq i \leq N-2 \end{aligned} \tag{2}$$

where μ is a free parameter. For N odd q is a primitive Nth or $2N$th root of unity while for N even q is restricted to primitive $2N$th roots of unity. Let us denote the representation of any $g \in \mathrm{U}_q$ by $\pi(g)$; we now solve, in the case $N=2$ and $N=3$, for a matrix $R \in \mathrm{End}(V \otimes V)$ such that

$$R(\pi \otimes \pi)\Delta(g) = (\pi \otimes \pi)\bar{\Delta}(g) R \qquad \text{all } g \in \mathrm{U}_q(\mathrm{sl}(2, \mathbb{C})) \tag{3}$$

where $\bar{\Delta}(g) = \tau\Delta(g)$ and $\tau(g \otimes g') = g' \otimes g$ for g, $g' \in \mathrm{U}_q$. We express our results in terms of the matrix $S = Pr$ where $P \in \mathrm{End}(V \otimes V)$ is the transposition map ($P(x \otimes y) = y \otimes x$, x, $y \in V$. Denoting by $S_{(2)}$ the solution corresponding to the two-dimensional case where $q^4 = 1$ and making the identification $p \equiv q^{\mu+1}$ we get

$$S_{(2)} = \text{block diag}(\sigma_1, \sigma_2, \sigma_{-1})$$

$$\sigma_1 = 1 \qquad \sigma_2 = \begin{pmatrix} 0 & p \\ p & 1-p^2 \end{pmatrix} \qquad \sigma_{-1} = p^2 q^{-2}. \tag{4}$$

In the case $N=3$ ($q^3 = 1$ or $q^6 = 1$) and with the identification $p \equiv q^{\mu+2}$ we obtain

$$S_{(3)} = \text{block diag}\,(\sigma_1, \sigma_2, \sigma_3, \sigma_{-2}, \sigma_{-1})$$

$$\begin{aligned} &\sigma_1 = 1 \qquad \sigma_{-1} = p^4 q^{-8} \\ &\sigma_2 = \begin{pmatrix} 0 & p \\ p & 1-p^2 \end{pmatrix} \qquad \sigma_{-2} = \begin{pmatrix} 0 & p^3 q^{-4} \\ p^3 q^{-4} & p^2(q^{-2} + q^{-4})(1 - p^2 q^{-2}) \end{pmatrix} \\ &\sigma_3 = \begin{bmatrix} 0 & 0 & p^2 \\ 0 & p^2 q^{-2} & -q^{-2}[p^2(1-p^2)(1-p^2q^{-2})(q^2+q^4)]^{1/2} \\ p^2 & -q^{-2}[p^2(1-p^2)(1-p^2q^{-2})(q^2+q^4)]^{1/2} & (1-p^2)(1-q^{-2}p^2) \end{bmatrix}. \end{aligned} \tag{5}$$

The matrix S satisfies

$$(S \otimes I)(I \otimes S)(S \otimes I) = (I \otimes S)(S \otimes I)(I \otimes S) \tag{6}$$

where $I \in \mathrm{End}(V)$ is the identity matrix. The relation (6) corresponds to one of the defining relations of Artin's braid group $\mathbb{B}_n$. $S_{(2)}$ and $S_{(3)}$ are the first two of a family of solutions that were first obtained by solving (6) directly [6]. These solutions proved to be of interest in knot theory where $S_{(2)}$ has been related to the Conway polynomial [7]. The connection between the Conway polynomial and the two-dimensional representation given in (2) is believed to be new. Note that the solution $\bar{S}_{(2)}$ obtained by setting $\sigma_{-1} = 1$ in (4) is related to the Jones polynomial and is associated with the two-dimensional representation of $\mathrm{U}_q(\mathrm{sl}(2, \mathbb{C}))$ with q generic.

The quantum R matrix $\check{R}(x)$ is a solution of the quantum Yang–Baxter equation

$$(\check{R}(x) \otimes I)(I \otimes \check{R}(xy))(\check{R}(y) \otimes I) = (I \otimes \check{R}(y))(\check{R}(xy) \otimes I)(I \otimes \check{R}(x)) \tag{7}$$

where $\check{R}(x) \in \mathrm{End}(V \otimes V)$ and x, y are the multiplicative spectral parameters. In order to transform a solution S of (6) into its corresponding quantum R matrix we use the following two formulae [8]

$$\check{R}(x) = S + \lambda_1 \lambda_2 x S^{-1} \tag{8}$$

in the case where S has two distinct eigenvalues λ_1 and λ_2, and

$$\check{R}(x)=\lambda_1\lambda_3 x(x-1)S^{-1}+\lambda_3\left(1+\frac{\lambda_1}{\lambda_2}+\frac{\lambda_1}{\lambda_3}+\frac{\lambda_2}{\lambda_3}\right)xI-(x-1)S \tag{9}$$

in the cases where S has three distinct eigenvalues λ_1, λ_2 and λ_3; I is the identity matrix. Note that in the both cases $\check{R}(x=0)=S$. While formula (8) is a proven result, formula (9) is only a conjecture which has proved to be valid in all cases examined so far; in this paper the validity of the results cited below have been established by direct substitution in (7) using a symbolic manipulation computer code. For details on how these formulae were constructed and the many cases in which they were used see [8–10]. Let us first consider the Baxterization of $S_{(2)}$ which has two distinct eigenvalues $\lambda_1=1$ and $\lambda_2=-p^2$. Substituting (4) into (8) and with the identification $x\equiv e^{-2\theta}$ and $p\equiv e^{\eta}$ we get

$$\begin{aligned}
&\check{R}(x)=\text{block diag }(\sigma_1,\sigma_2,\sigma_{-1})\\
&\sigma_1=\sinh(\eta-\theta)\qquad \sigma_{-1}=\sinh(\eta+\theta)\\
&\sigma_2=\begin{pmatrix}\sinh(\eta) & \sinh(\theta)\\ \sinh(\theta) & \sinh(\eta)\end{pmatrix}
\end{aligned}\tag{10}$$

where we made use of the symmetry-breaking transformation described in [11] to restore the symmetry of the diagonal elements; (10) is the quantum R matrix of the free fermion model of Fan and Wu [4].

The Baxterization of $S_{(3)}$ is done using (9) with $\lambda_1=1$, $\lambda_2=p-p^2$ and $\lambda_3=p^4q^{-2}$

$$\begin{aligned}
&\check{R}(x)=\text{block diag}(\sigma_1,\sigma_2,\sigma_3,\sigma_{-2},\sigma_{-1})\\
&\sigma_1=p^4q^4x(x-1)+p^4q^4x(1-q^2p^{-2})(1-p^{-2})-(x-1)\\
&\sigma_{-1}=x(x-1)+p^4q^4(1-q^2p^{-2})(1-p^{-2})x-p^4q^4(x-1)\\
&\sigma_{-2}=\begin{pmatrix}p^2q^4x(1-p^{-2}q^2)(p^2-x) & pq^2(x-1)(x-p^2)\\ pq^2(x-1)(x-p^2) & p^4q^4(1-p^{-2}q^2)(1-p^{-2}x)\end{pmatrix}\\
&\sigma_2=\begin{pmatrix}xp^4q^4(1-p^{-2})(x-q^2p^{-2}) & p^3q^4(x-1)(x-p^{-2}q^2)\\ p^3q^4(x-1)(x-p^{-2}q^2) & p^4q^4(1-p^{-2})(x-q^2p^{-2})\end{pmatrix}\\
&\sigma_3=\begin{pmatrix}p^4q^4x^2(1-p^{-2})(1-p^{-2}q^2) & -q^4\alpha x(x-1) & p^2q^4(x-1)(x-q^2)\\ -q^4\alpha x(x-1) & \beta & -q^4\alpha(x-1)\\ p^2q^4(x-1)(x-q^2) & -q^4\alpha(x-1) & (1-p^{-2}q^2)(1-p^{-2})p^4q^4\end{pmatrix}
\end{aligned}\tag{11}$$

with

$$\begin{aligned}
&\alpha\equiv[p^2(1-p^2)(1-p^2q^{-2})(q^2+q^4)]^{1/2}\\
&\beta\equiv p^2x(x-1)+p^4q^4(1-q^2p^{-2})(1-p^{-2})x-p^2q^4(x-1)
\end{aligned}$$

and where $q^3=1$ or $q^6=1$.

We conclude by briefly discussing related works.

In this paper we have linked the solutions $S_{(2)}$ and $S_{(3)}$ reported in [6] to representations of U_q (sl(2, $\mathbb{C}$)) when $q^m=1$. Recently, other interpretations of these solutions have been proposed. In [12], the following two-parameters (p and q) deformation of the enveloping algebra of A_1, to which an additional generator H_0 has been added, has been proposed.

$$\begin{aligned}
&[H_1,Z^{\pm}]=\pm 2Z^{\pm}\qquad Z^+Z^--p^2Z^-Z^+=\frac{k_2^2-k_1^{-2}}{q-q^{-1}}\\
&k_1Z^{\pm}k_1^{-1}=(qp)^{\pm}Z^{\pm}\qquad k_2Z^{\pm}k_2^{-1}=(qp)^{\pm}p^{\mp 2}Z^{\pm}
\end{aligned}\tag{12}$$

with

$$k_1 = (qp)^{H_1/2} \qquad k_2 = p^{H_0}(q/p)^{H_1/2} \tag{13}$$

and where H_0 commutes with all generators. We omit the details of the Hopf structure. When $p = 1$, we have $k_1 = k_2$ and the above algebraic structure reduces to that of $U_q(\mathrm{sl}(2, \mathbb{C}))$. The interesting case is when q is equal to a root of unity and p is generic: using $\mathcal{R} = \Sigma_\sigma e_\sigma \otimes e^\sigma$ as his definition of universal $\mathcal{R}$ matrix, with e_σ being the basis of the Hopf subalgebra generated by H_0, H_1 and Z^-, and e^σ its dual basis, Lee gives N-dimensional representations of the generators H_0, H_1, Z^+ and Z^- and explicit bases e_σ and e^σ; in this way he generates a series of solutions of (6) the first two of which correspond to (4) and (5). He views this algebra as different from $U_q(\mathrm{sl}(2, \mathbb{C}))$ (it has two k's (k_1 and k_2), different relations and Hopf structure and an additional generator) and refers to it as the twisted quantum group of A_1. The obvious question is whether there is any connection between the algebraic structure described in (13) and that of $U_q(\mathrm{sl}(2, \mathbb{C}))$. If we define a new set of generators J^+, J^- and k such that

$$J^+ \equiv k_1 Z^+ \qquad J^- \equiv Z^- k_2^{-1} \qquad k^2 = k_1 k_2$$

it is easy to check that the algebra defined in (12) reduces to $U_q(\mathrm{sl}(2, \mathbb{C}))$ with the only difference being in the definition of k:

$$k = q^{H_1/2} p^{H_0/2}.$$

Following such a transformation p no longer appears as a deformation parameter, its only role being in the definition of k. The twisted quantum group of A_1 as defined in [12] can now be viewed as $U_q(\mathrm{sl}(2, \mathbb{C}))$ at roots of unity; the presence of the parameter p is consistent with the fact that the representations described in (2) are parametrized by one free parameter μ which can be recovered by setting $p \equiv q^\mu$; finally, H_1 and H_0 can be combined into a single generator.

The quantized universal enveloping algebra associated with the solutions reported in [6] was also investigated in [13, 14], using a method [15] of Fadeev *et al* which consists in constructing the algebra starting with the matrix S. Although the method has been applied only to $S_{(2)}$ [13] and $S_{(3)}$ [14], the main result of this exercise points to the following conclusion: starting with $S_{(N)}$ one obtains an algebra whose relations are those of $U_q(\mathrm{sl}(2, \mathbb{C}))$ with $q = \omega_N$ ($\omega_N^N = 1$) and with $(J^+)^N = (J^-)^N = 0$ as additional relations; we denote this algebra $U_{\omega_N}(\mathrm{sl}(2, \mathbb{C}))$. $U_{\omega_N}(\mathrm{sl}(2, \mathbb{C}))$ has only N-dimensional faithful irreducible representations. To every solution $S_{(N)}$ one therefore associates a different algebra (different in the sense that the value of q and the additional relations are different). These results are consistent with the results presented in this letter and reflect the fact that these solutions are associated with the two- and three-dimensional representations at the root of unity. Recently, $S_{(2)}$ and $\bar{S}_{(2)}$ have been related through the Lie superalgebra $\mathrm{gl}(K/L)$ with $K + L = 2$ [16].

Finally, in a recent letter [17] it was shown that one may obtain solutions such as $S_{(2)}$ and $S_{(3)}$ by restricting a modified universal R-matrix to the two- and three-dimensional representations of $U_q(\mathrm{sl}(2, \mathbb{C}))$ for q not a root of unity. In summary we have examined in this letter a method of constructing quantum R matrices associated with representations of $U_q(\mathrm{sl}(2, \mathbb{C}))$ at roots of unity via Baxterization. We have considered certain highest-weight representations proposed by Roche and Arnaudon [5] and more recently by De Concini and Kac [18], and in the process have shown that the solutions reported in [6] are associated with these representations. The method could be applied in principle to any quantized enveloping algebra; progress in this direction will require proofs of the Baxterization formulae for three and more distinct eigenvalues (to our knowledge a proof does not exist).

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